Guide to the Herschel Gyro-based Attitude Reconstruction Software

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1 Introduction

Despite the success achieved during the mission in improving the accuracy of the attitude measurements made by the star tracker [6, 21, 22], and later the success in further improving these measurements on-ground [7], it has been shown that the on-board attitude filter is still rather poor at following the high-frequency changes in the spacecraft attitude (i.e. the spacecraft jitter) [2, 3].[1] A new method, the 'gyro-based attitude reconstruction' method, was therefore proposed for combining the star tracker attitude measurements with the output from the Inertial Reference Unit (GYR) [8]. As the name implies, this method places much greater weight on the measurements made by the gyros, using attitude measurements constructed from star tracker data to provide an absolute reference and to account for gyro drifts. Although it has been demonstrated that this method is able to successfully reduce the high-frequency components of the Absolute Measurement Error (AME) it neverthess suffers from a couple of drawbacks (when compared with a conventional attitude estimator). When 'calibrating' the gyro measurements, firstly, only those attitude measurements close to the reference attitude may be used and, secondly, a deterministic model of the variation of the gyro drift rates must be assumed.[2]

Software based on the gyro-based attitude reconstruction method was first prototyped by H. Feuchtgruber and later implemented, in HIPE 12.0, by B. Vandenbussche. The author's involvement has been with its further refinement for the release of HIPE 13.0. In addition to describing how to use the software (Section 3), this document provides a detailed description of the method which has been implemented (Section 2). A (non-exhaustive) list of the issues known to be currently affecting the software is included (Section 4).

[1] The consequence of improved star tracker attitude measurements is a reduction in the bias and low-frequency components of the Absolute Measurement Error (AME). When the AME is reduced on-board it has the additional effect of simultaneously reducing the Absolute Pointing Error (APE).

[2] The software currently assumes the gyro drift rates to remain constant over intervals of a fixed length (see p. 16, footnote 32). A more general-purpose estimator, in which low-pass filtered attitude measurements from the star tracker are combined with high-pass filtered gyro measurements, has since been prototyped and shown to produce equally-accurate results for the case of staring observations [22]. Furthermore, M. Tuttlebee has shown that, by simply retuning the gains, the on-board filter may be used to produce similar results. Whether similar results can be achieved for other observation types requires further investigation.

2 Method

2.1 Improvement of star tracker attitude measurements

Although not truly part of the gyro-based attitude reconstruction method, the first function of the software is to use the data contained in the STR-specific diagnostics telemetry dataset (`AcmsDtmStr`), whenever these data are present, to create a set of improved (often referred to as 'corrected') star tracker attitude measurements.[3] This dataset contains at each timestep (typically every 1 s, but every 0.25 s is also possible) the following information for each of the (up to) nine stars tracked by the star tracker:[4]

- y_b- and z_b-coordinates, u'_{y_b} and u'_{z_b}, of the aberration-corrected, measured star vector in the Boresight Reference Frame (BRF);
- Catalogue ID of the matched star;
- right ascension and declination of the matched star (corrected for proper motion);
- parameter, α_c, used to correct the star tracker focal length according to the colour of the star; see (4).

Using these data and the residual distortion maps [see 7], the software constructs improved measurements of the star directions and these in turn are used to create improved attitude measurements (replacing those that were originally produced on-board by the star tracker).

Before explaining in detail how the improved attitude measurements are created, it is helpful to review the procedure employed on-board the spacecraft to convert the measured CCD coordinates of each star into an aberration-corrected, measured star vector, $\mathbf{u}'$.

[3]In this document the term 'star tracker attitude measurement' refers to an estimate of the spacecraft attitude, i.e. of the Attitude Control Axes (ACA) reference frame, that is made using data downlinked from the star tracker. It does not refer to a measurement of the star tracker attitude, i.e. of the Boresight Reference Frame (BRF).

[4]Other information, not used by the software, is also available in this dataset [see 15, pp. 39–45]. When interlacing moded is active, the star tracker may track up to 18 stars, but this functionality is not handled by the software.

2.1.1 On-board construction of measured star vectors

The process which was used on-board by the star tracker to convert the measured CCD coordinates of each star into an aberration-corrected, measured star vector can be broken down into three steps:

1. *Correction for star tracker CCD distortion*

   Let the position of the star on the CCD, as determined by the centroiding algorithm, be (y, z).[5] The distortion-corrected coordinates, (y', z'), were calculated on-board according to:[6]

$$\begin{aligned} y' &= F(y, z; k_0, \ldots, k_7), \\ z' &= F(z, y; h_0, \ldots, h_7), \end{aligned} \tag{1}$$

   where

$$\begin{aligned} F(u, v; a_0, \ldots, a_7) = a_0 &+ a_1 u + a_2 v + a_3 u(u^2 + v^2) + a_4 u(u^2 + v^2)^2 \\ &+ a_5 u^2 + a_6 uv + a_7 v^2, \end{aligned} \tag{2}$$

   and the a_i $(i = 0, \ldots, 7)$ are real constants. The coefficients k_0, k_1, $\ldots$, k_7 and h_0, h_1, $\ldots$, h_7 are referred to as the 'distortion correction coefficients' for the y- and the z-axis respectively.[7]

2. *Calculation of the focal length and apparent star direction*

   The apparent direction of the star with respect to the BRF is given by the unit vector:[8]

$$\mathbf{u} = \begin{pmatrix} f \\ -y' \\ -z' \end{pmatrix} (f^2 + y'^2 + z'^2)^{-\frac{1}{2}}, \tag{3}$$

[5] The variables y and z correspond respectively to the variables y_{fcd} and z_{fcd} in [4, p. 83].

[6] Eqs. (1) and (2) correspond precisely to eqs. (9.2-3) in [4, p. 83], with y' and z' replacing y and z, and y and z replacing y_{fcd} and z_{fcd}.

[7] Three sets of values for the parameters k_i and h_i were used on-board during the mission, depending on the Operational Day (OD): (i) OD $\in \{1, 2, \ldots, 865\} \setminus \{858\}$: pre-launch values; (ii) OD $\in \{858\} \cup \{866, 867, \ldots, 1010\} \setminus \{1005\}$: modified values of k_1 and h_1; (iii) OD $\in \{1005\} \cup \{1011, 1012, \ldots, 1446\}$: all values modified.

[8] Eq. (3) corresponds precisely to eqs. (9.2-5) in [4, p. 83], with y' and z' replacing y and z.

where the focal length, f, of the star tracker is determined from:[9]

$$f \equiv f(f_0, \alpha_c) = f_0[1 + \alpha_T(T - T_0) + \alpha_c]. \quad (4)$$

f_0 represents the nominal focal length of the star tracker at the reference temperature, $T_0 = 22$ C, and the parameter α_c (as obtained from the entry of the matched star in the on-board star catalogue) is used to correct the nominal focal length according to the colour of the star.[10] For the entire duration of the mission, the coefficient of thermal expansion of the objective, α_T, was set equal to 5.91×10^{-6} C^{-1} and the temperature, T, of the star tracker optics (which may found in the housekeeping telemetry) was found to remain constant at 13.73 C.[11]

3. *Correction of the stellar aberration*

   Stellar aberration is corrected using the classical model [5]. That is, the aberration-corrected (unit) vector in the direction of the star is given by:

$$\mathbf{u}' = \mathbf{u} + \mathbf{u} \times (\mathbf{u} \times \mathbf{v}_{s/c})/c, \quad (5)$$

   where $\mathbf{v}_{s/c}$ is the spacecraft velocity vector and c is the speed of light.[12]

2.1.2 On-ground processing

The process by which the attitude reconstruction software constructs, at each timestep (i.e. every 1 s), an improved attitude measurement is the following:

Removal of the on-board distortion correction

The first step in the process is to take each of the (aberration-corrected) measured star vectors

$$\mathbf{u}' = \begin{pmatrix} \sqrt{1 - (u'_{y_b})^2 - (u'_{z_b})^2} \\ u'_{y_b} \\ u'_{z_b} \end{pmatrix}, \quad (6)$$

[9]See eq. (9.2-1) of [4, p. 82].

[10]Two values for the nominal focal length, f_0, were used on-board during the mission: $f_0 = 29.96330$ mm (OD < 762), $f_0 = 29.97178$ mm (otherwise).

[11]See [7, p. 6].

[12]Since $\mathbf{u}'$ is required with respect to the BRF, the spacecraft velocity vector, $\mathbf{v}_{s/c}$, is transformed from inertial coordinates to BRF before using eq. (5).

(expressed here in the BRF) and to remove the distortion correction that was applied on-board, thereby recovering the measured position of the star on the CCD. This is performed by:

1. Removing the aberration correction:

$$\mathbf{u} = \mathbf{u}' - \mathbf{u}' \times (\mathbf{u}' \times \mathbf{v}_{\mathrm{s/c}})/c. \tag{7}$$

2. Calculating the distortion-corrected CCD coordinates of the star:[13]

$$\begin{aligned} y'_{\mathrm{r}} &= f\frac{u_y}{u_x}, \\ z'_{\mathrm{r}} &= f\frac{u_z}{u_x}, \end{aligned} \tag{8}$$

   where u_x, u_y and u_z are the three components of $\mathbf{u}$ expressed in the BRF, and the focal length, f, is calculated from (4) using the value of f_0 that was on-board during the OD in question (see footnote 10) and the value of α_{c} contained in `AcmsDtmStr`.[14]

3. Solving

$$\begin{aligned} y'_{\mathrm{r}} &= F_1(y_{\mathrm{r}}, z_{\mathrm{r}}; k_0, \ldots, k_7), \\ z'_{\mathrm{r}} &= F_1(z_{\mathrm{r}}, y_{\mathrm{r}}; h_0, \ldots, h_7), \end{aligned} \tag{9}$$

   where

$$\begin{aligned} F_1(u, v; a_0, \ldots, a_7) =& - F(-u, -v; a_0, \ldots, a_7) \\ =& - a_0 + a_1 u + a_2 v + a_3 u(u^2 + v^2) + a_4 u(u^2 + v^2)^2 \\ & - a_5 u^2 - a_6 uv - a_7 v^2, \end{aligned} \tag{10}$$

   and the coefficients k_i and h_i are those that were on-board during the OD in question (see footnote 7), to obtain the measured position $(y_{\mathrm{r}}, z_{\mathrm{r}})$ of the star on the CCD.

[13] The subscript ‘r’ has been used to distinguish the variables used in the attitude *reconstruction* software from those used on-board.

[14] Note that eqs. (9.3-2) of [4, p. 84] are incorrect; in each a factor $-f$ is required for consistency with eqs. (9.2-5).

The software computes an approximate solution of (9) by means of

$$\begin{aligned} y_{\mathrm{r}} &= F_1(y'_{\mathrm{r}}, z'_{\mathrm{r}}; k'_0, \ldots, k'_7), \\ z_{\mathrm{r}} &= F_1(z'_{\mathrm{r}}, y'_{\mathrm{r}}; h'_0, \ldots, h'_7), \end{aligned} \tag{11}$$

where the coefficients k'_i and h'_i of these 'inverse polynomials' have been computed so as to remove the distortion corrections applied by (9) [see 7, p. 7 and p. 25]. Despite the seeming lack of theoretical justification for using polynomials of this form, it has been demonstrated (and indepedently verified) that, within the region of the CCD used to track guide stars (a disc of radius 4.07 mm), the errors introduced by this method are all less than $0.05''$ [7, p. 7]. An alternative approach is described in Appendix A.

Setting $y'_{\mathrm{r}} = -y'$ and $z'_{\mathrm{r}} = -z'$ in (8) we obtain equations which are consistent with (3) and then setting $y_{\mathrm{r}} = -y$, $z_{\mathrm{r}} = -z$ in (9) we recover (1), showing that the CCD coordinates used by the attitude improvement software are of opposite sign to those used in eq. (9.2-3) of [4, p. 83].[15] Although it is unclear precisely why it was decided to use coordinates of opposite sign,[16] this is of no consequence, provided that consistent transformations are used when re-applying the distortion correction (12) and converting back to a unit vector (13).

Application of the new distortion correction

Having removed the on-board distortion correction and recovered the measured position of the star on the CCD, the new distortion correction may be applied. The new correction consists of replacing (9) by:

$$\begin{aligned} y'_{\mathrm{r}} &= F_1(y_{\mathrm{r}} - \widetilde{\Delta y}_{\mathrm{r}}, z_{\mathrm{r}} - \widetilde{\Delta z}_{\mathrm{r}}; k_0, \ldots, k_7), \\ z'_{\mathrm{r}} &= F_1(z_{\mathrm{r}} - \widetilde{\Delta z}_{\mathrm{r}}, y_{\mathrm{r}} - \widetilde{\Delta y}_{\mathrm{r}}; h_0, \ldots, h_7), \end{aligned} \tag{12}$$

[15] Alternatively, from (3) and (8) we obtain $y'_{\mathrm{r}} = -y'$, $z'_{\mathrm{r}} = -z'$ and then using (9) and (10):

$$\begin{aligned} y' &= F(-y_{\mathrm{r}}, -z_{\mathrm{r}}; k_0, \ldots, k_7), \\ z' &= F(-z_{\mathrm{r}}, -y_{\mathrm{r}}; h_0, \ldots, h_7), \end{aligned}$$

from which, upon comparison with (1), gives $y_{\mathrm{r}} = -y$, $z_{\mathrm{r}} = -z$.

[16] Perhaps some confusion arose from a failure to notice the sign error in (9.3-2) of [4, p. 84] (see footnote 14), followed by an attempt to compensate for this error by changing the sign of all the terms of even degree in the expression for F, i.e. in (2).

where the values of the distortion correction coefficients, $k_0, \ldots, k_7, h_0, \ldots, h_7$ are those that were on-board between ODs 320–762 (a period which is referred to as the 'reference period' [7, p. 7]).[17]

The 'residual distortions', Δy_r and Δz_r, are obtained from a pair of maps (one for each axis).[18] Each map consists of a 5120×5120 array, where each element of the array is identified with a sub-pixel of the CCD (each pixel having been divided uniformly into $10 \times 10 = 100$ sub-pixels).

The residual distortions have been shown to comprise both local (due to the location of the sub-pixel within the pixel) and global phenomena. To reduce the errors present in the local corrections, without unduly affecting the global corrections, the following averaging is employed. Suppose the star is found to lie in a particular sub-pixel of pixel (m, n), where $0 \leq m, n \leq 511$. Then $\widetilde{\Delta y}$ and $\widetilde{\Delta z}$ are taken to be the median values of the residual distortions at the corresponding sub-pixel of the 121 pixels centred on pixel (m, n), i.e. from pixels $\{(i, j) : m - 5 \leq i \leq m + 5, n - 5 \leq j \leq n + 5\}$.[19]

Having applied the new distortion correction, it remains simply to construct the star (unit) vector,

$$\mathbf{u} = \begin{pmatrix} f \\ y_\mathrm{r}' \\ z_\mathrm{r}' \end{pmatrix} (f^2 + {y_\mathrm{r}'}^2 + {z_\mathrm{r}'}^2)^{-\frac{1}{2}}, \tag{13}$$

where the focal length f is calculated using (4), making sure to use the value of f_0 that was on-board during the reference period,[20] and then to correct for stellar aberration using (5).

[17] Although a justification for the decision to use this period as the reference period is given in [7, p. 7], the choice was, in fact, quite arbitrary. The only point of importance is that values of the distortion correction coefficients and of f_0 should be the same as those that were used when producing the residual distortion maps.

[18] There are two such pairs of maps: one for the period prior to OD 320 (when the set-point for the temperature of the star tracker CCD was reduced to -10 C), and one for the remaining ODs. It is assumed that during each of these periods the distortion of the CCD remained constant.

[19] Note that this is not the method described in [7, p. 12]. Note also that the software includes checks to ensure that the centroid of the star and all the pixels used for this averaging lie within the operational portion (disc) of the CCD plane.

[20] The decision to use this value was once again quite arbitrary.

Estimation of the star tracker attitude

From the previous two steps we have obtained a set of up to nine measured star vectors $\{\mathbf{u}'_1, \mathbf{u}'_2, \ldots, \mathbf{u}'_n\}$, $n \leq 9$, each in the BRF and each with the new distortion correction applied. (In fact, only stars whose centroid lies inside the operational region of the field-of-view are used for the attitude determination. This is a disc, centred on the image of the boresight, of radius 4.07 mm. That is, all star vectors must be within 7.7° of the boresight direction.) The corresponding (expected) inertial directions $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ are then easily derived from information which may be extracted from the star catalogue using the IDs of the matched stars.

The improved estimate of the spacecraft attitude is taken to be the rotation matrix $A \in \mathrm{SO}(3)$, i.e. the three-dimensional orthogonal matrix with determinant 1, which minimizes the loss (cost) function

$$L(A) \equiv \frac{1}{2}\sum_{i=1}^{n} a_i |\mathbf{u}'_i - A\mathbf{v}_i|^2, \qquad n \geq 2, \tag{14}$$

where the a_i are a set of non-negative weights.[21] If one adopts the QUEST measurement model [20], that is, if one assumes that each measured star vector, $\mathbf{u}'_i$, is related to its true direction, $\mathbf{u}'_{i,\mathrm{true}}$, by

$$\mathbf{u}'_i = \mathbf{u}'_{i,\mathrm{true}} + \Delta\mathbf{u}'_i,$$

where the measurement errors, $\Delta\mathbf{u}'_i$, are mutually independent and normally distributed random variables

$$\Delta\mathbf{u}'_i \sim N(\mathbf{0}, R_i),$$

with covariance matrices

$$R_i = \sigma_i^2 (I_{3\times3} - \mathbf{u}'_{i,\mathrm{true}}\mathbf{u}'^{\mathrm{T}}_{i,\mathrm{true}}), \tag{15}$$

and if, furthermore, one chooses the weights such that

$$a_i = \frac{c}{\sigma_i^2}, \tag{16}$$

where c is some arbitrary constant, then it has been shown [see 17] that the attitude matrix, A^*, which minimizes the loss function $L(A)$ is also the

[21] This problem was first posed, in an unweighted form, by Wahba [23].

maximum likelihood attitude estimate. Equation (15) is equivalent to assuming that the error $\Delta\mathbf{u}'_i$ is perpendicular to $\mathbf{u}'_{i,\mathrm{true}}$ and that it has an axially-symmetric distribution with variance σ_i^2.

To minimize the loss function, Davenport's q-method is employed. First, $L(A)$ is rewritten [e.g. 12, p. 360] in terms of the gain function g as:

$$L(A) = \lambda_0 - g(A), \tag{17}$$

where $\lambda_0 = \sum_{i=1}^{n} a_i$,

$$g(A) = \mathrm{tr}(AB^{\mathrm{T}}),$$

and

$$B = \sum_{i=1}^{n} a_i \mathbf{u}'_i \mathbf{v}_i^{\mathrm{T}}.$$

So, to minimize $L(A)$ we must maximize $g(A)$. In Davenport's method, the loss function is re-written in terms of the attitude quaternion $\mathbf{q}$, i.e.

$$\tilde{L}(\mathbf{q}) = \lambda_0 - \tilde{g}(\mathbf{q}), \tag{18}$$

where $\tilde{L}(\mathbf{q}) = L(A(\mathbf{q}))$ and $\tilde{g}(\mathbf{q}) = g(A(\mathbf{q}))$, and it is shown that

$$\tilde{g}(\mathbf{q}) = \mathbf{q}^{\mathrm{T}} K \mathbf{q}, \tag{19}$$

where

$$K = \begin{bmatrix} S - I\,\mathrm{tr}(B) & \mathbf{z} \\ \mathbf{z}^{\mathrm{T}} & \mathrm{tr}(B) \end{bmatrix},$$

$S = B + B^{\mathrm{T}}$ and $\mathbf{z} = \begin{bmatrix} B_{23} - B_{32} \\ B_{31} - B_{13} \\ B_{12} - B_{21} \end{bmatrix}$.

Applying Lagrange's "method of the undetermined multiplier" to the problem of maximizing (19) subject to the constraint $\mathbf{q}^{\mathrm{T}}\,\mathbf{q} = 1$, it is easy to see that our optimal attitude estimate, $\widehat{\mathbf{q}}$, is an eigenvector of K and from (19) it is also clear that it corresponds to the maximum eigenvalue, $\lambda_{\max}$.

Having solved the eigenvalue problem and obtained $\widehat{\mathbf{q}}$, it remains simply to multiply $\widehat{\mathbf{q}}$ by the appropriate star tracker alignment quaternion, $\mathbf{q}_{\mathrm{align}}$, in order to obtain an estimate, $\widehat{\mathbf{q}}_{\mathrm{str}}$, of the ACA-frame attitude based on star tracker data.[22] That is,

$$\widehat{\mathbf{q}}_{\mathrm{str}} = \widehat{\mathbf{q}}\, \mathbf{q}_{\mathrm{align}}. \tag{20}$$

Calculation of goodness-of-fit

Now the value of c in (16) is arbitrary and in the pointing reconstruction software we have set $c = (\sum_{i=1}^{n} 1/\sigma_i^2)^{-1}$ so as to make $\lambda_0 = 1$. Furthermore, we have chosen to weight all the measurements equally, i.e. $\sigma_i^2 = \sigma^2$, so that $a_i = 1/n$ $(i = 1, \ldots, n)$.

In [19] Shuster defines the variable TASTE (for the case where $c = 1$, $a_i = 1/\sigma_i^2$) and shows it to be distributed randomly according to a χ^2-distribution with $2n - 3$ degrees of freedom.[23] In [18] he provides a more general definition:

$$\mathrm{TASTE} = \frac{2L(A^*)}{\lambda_0\, \sigma_{\mathrm{tot}}^2} = \frac{2(\lambda_0 - \lambda_{\max})}{\lambda_0\, \sigma_{\mathrm{tot}}^2}, \tag{21}$$

where $\sigma_{\mathrm{tot}}^2 = (\sum_{i=1}^{n} 1/\sigma_i^2)^{-1}$, which makes the result independent of the choice for c (or equivalently λ_0).

For the value of c chosen above, we find that

$$\mathrm{TASTE} = \frac{2n\, \tilde{L}(\widehat{\mathbf{q}})}{\sigma^2} = \frac{2n\,(1 - \lambda_{\max})}{\sigma^2}. \tag{22}$$

This is the expression for TASTE which is used in the pointing reconstruction software.

To estimate σ we may use the procedure suggested in [19, pp. 5–8]. Suppose that for a given observation there are N time steps and that at time step i the attitude is estimated from n_i star vectors and that the resulting

[22]For most of the mission—other than immediately following a star tracker reset—the operational star tracker was STR-A. The alignment quaternion for this star tracker is: $\mathbf{q}_{\mathrm{align}} = \begin{bmatrix} 0.1953842 \times 10^{-3} & -0.2993422 \times 10^{-1} & -0.9995519 & 0.2061553 \times 10^{-4} \end{bmatrix}^{\mathrm{T}}$, which amounts to a rotation of approximately 180° through an axis close to $z_{\text{str-a}}$ [1, p. 77 and p. 602]. Throughout this document, the quaternion $q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} + q_4$ will be represented by the 4-vector $\begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^{\mathrm{T}}$.

[23]This result, which seems very familiar, is normally only true for least-squares problems in which the model is linear with respect to the fitted parameters [14, p. 654]. Wahba's problem also differs from a standard least-squares problem in that the parameters are constrained by the attitude belonging to SO(3).

value of TASTE, when σ is set equal to σ_{ref} in (22), is TASTE_i. Then our estimate of σ is:

$$\widehat{\sigma} = \sqrt{\frac{\sigma_{\text{ref}}^2 \sum_{i=1}^{N} \text{TASTE}_i}{2\sum_{i=1}^{N} n_i - 3N}}. \tag{23}$$

Based on a handful of observations, the author has found $\widehat{\sigma} \approx 2.9''$.[24]

The variable TASTE, calculated according to (22), provides an indication of the goodness-of-fit: the larger the value of TASTE, the worse the fit. In fact, if our measurement model is accurate, we would expect:

$$p_{\text{taste}} \equiv \Pr\{\text{TASTE} > val\} = 1 - P\left(\frac{2n-3}{2}, \frac{val}{2}\right), \tag{24}$$

where P is the lower (regularized) incomplete gamma function. If the value of p_{taste} is found to be below a certain threshold—which may be specified by means of the parameter `prob_thresh` (see p. 35, Table 2)—then the fit is considered to be sufficiently bad as to indicate the presence of an outlier. The software then attempts to remove star vector measurements until p_{taste} is above this threshold.[25]

Finally, if we follow Shuster and Oh [20, p. 71] and define the Cartesian attitude covariance matrix as

$$P_{\theta\theta} = \text{E}(\delta\theta\,\delta\theta^{\text{T}}),$$

where $\delta\theta = \begin{pmatrix}\delta\theta_1 & \delta\theta_2 & \delta\theta_3\end{pmatrix}^{\text{T}}$ and the $\delta\theta_i$ are the small-angle rotations about the body (ACA-frame) axes which take the true spacecraft attitude to its estimated attitude, then

$$P_{\theta\theta} = \sigma^2 A_{\text{align}} \left[\sum_{i=1}^{n} (I - \mathbf{u}_i' \, \mathbf{u'}_i^{\text{T}})\right]^{-1} A_{\text{align}}^{\text{T}}, \tag{25}$$

where $A_{\text{align}} = A(\mathbf{q}_{\text{align}})$.[26]

[24] This value is currently hard-coded in the pointing reconstruction sofware, i.e. assigned (in radians) to the parameter `sigmaVectorMeasurement`; see `HCSS-19267`.

[25] At present the software excludes a maximum of two measurements; this appears to be sufficient.

[26] The expression for the covariance matrix in the BRF follows from eqs. (87) and (99) of [20, p. 75], where it is also noted that a singular matrix $\sum_{i=1}^{n}(I - \mathbf{u}_i' \, \mathbf{u'}_i^{\text{T}})$ would signal a non-unique solution for the optimal quaternion $\widehat{\mathbf{q}}$.

2.2 Gyro-based attitude reconstruction

2.2.1 Derivation of the model

Suppose we wish to estimate the (ACA-frame) attitude of the spacecraft, $\mathbf{q}_{\rm aca}(t)$, at the time, t_k, of the k'th gyro measurement. We start by choosing a reference attitude $\mathbf{q}_{0,k}$ such that, for t sufficiently close to t_k, the rotation between $\mathbf{q}_{0,k}$ and $\mathbf{q}_{\rm aca}(t)$, i.e. the angular rotation represented by the quaternion

$$\mathbf{q}_{\rm r}^{(k)}(t) \equiv \mathbf{q}_{0,k}^{*}\, \mathbf{q}_{\rm aca}(t), \tag{26}$$

is small (* denotes the conjugate operation).[27]

Suppose $\mathbf{q}_{\rm r}^{(k)}(t)$ corresponds to the rotation $\theta^{(k)}(t)$ about the unit vector $\mathbf{e}^{(k)}(t)$. Dropping the explicit references to t and k, we may write

$$\mathbf{q}_{\rm r} = \begin{bmatrix} e_x \sin\frac{\theta}{2} & e_y \sin\frac{\theta}{2} & e_z \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}^{\rm T}, \tag{27}$$

where e_x, e_y and e_z are the components of $\mathbf{e}$ with respect to the $\mathbf{q}_{0,k}$ reference frame. For small θ, we may approximate $\sin\frac{\theta}{2}$ by $\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$ by 1, so that

$$\mathbf{q}_{\rm r} = \begin{bmatrix} \frac{\theta_x}{2} & \frac{\theta_y}{2} & \frac{\theta_z}{2} & 1 \end{bmatrix}^{\rm T} + \mathrm{O}\,(\theta^2), \tag{28}$$

where $\theta_x = \theta\, e_x$, $\theta_y = \theta\, e_y$ and $\theta_z = \theta\, e_z$, and the kinematic equations of motion [e.g. 24, p. 512]

$$\frac{d\mathbf{q}_{\rm r}}{dt} = \frac{1}{2}\,\Omega\,\mathbf{q}_{\rm r}, \quad \Omega = \begin{bmatrix} 0 & \omega_w & -\omega_v & \omega_u \\ -\omega_w & 0 & \omega_u & \omega_v \\ \omega_v & -\omega_u & 0 & \omega_w \\ -\omega_u & -\omega_v & -\omega_w & 0 \end{bmatrix},$$

[27] To achieve this, `calcGyroAttitude` sets $\mathbf{q}_{0,k} = \widehat{\mathbf{q}}_{\rm str}^{(m)}$, where $\widehat{\mathbf{q}}_{\rm str}^{(m)}$ is the estimate of the ACA-attitude based on the star tracker data at time $t_{\rm s}^{(m)}$ and $m = \max\{i : t_{\rm s}^{(i)} \le t_k\}$, whenever it is found that the rotation represented by $\mathbf{q}_{0,k}^{*}\, \widehat{\mathbf{q}}_{\rm str}^{(m)}$ exceeds a certain threshold, which the User may specify by means of the input parameter `ref_thresh` . In addition, when updating the reference attitude it is ensured that only good quality attitude measurements are used, although it is currently still possible that a bad quality measurement may be taken as the initial reference (see `HCSS-19775`).

become:

$$
\begin{aligned}
\dot{\theta}_x &= \omega_u + \frac{1}{2}[\omega_w\,\theta_y - \omega_v\,\theta_z] + \mathrm{O}\,(\theta^2),\\
\dot{\theta}_y &= \omega_v + \frac{1}{2}[\omega_u\,\theta_z - \omega_w\,\theta_x] + \mathrm{O}\,(\theta^2),\\
\dot{\theta}_z &= \omega_w + \frac{1}{2}[\omega_v\,\theta_x - \omega_u\,\theta_y] + \mathrm{O}\,(\theta^2),
\end{aligned}
\tag{29}
$$

where ω_u, ω_v and ω_w are the components of the angular velocity vector in the body-fixed (i.e. ACA) reference frame and clearly $\theta = \sqrt{\theta_x^2 + \theta_y^2 + \theta_z^2}$.

The determination of the body rates, ω_u, ω_v and ω_w, from the rates, ω_1, ω_2, ω_3 and ω_4, about each of the four gyro axes is an overdetermined problem. We may either disregard the measurements from one of the gyros, using them instead as a check on the quality of the measurements from the other three, or, as is currently performed in the pointing reconstruction software, find the least-squares solution.[28] That is,

$$
\begin{bmatrix}\omega_u\\ \omega_v\\ \omega_w\end{bmatrix} = G^{+}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}, \tag{30}
$$

where $G^{+} = (G^{\mathrm{T}}G)^{-1}G^{\mathrm{T}}$ is the Moore-Penrose pseudoinverse of the gyro alignment matrix G.[29]

Substituting (30) in (29) and neglecting terms of degree one in θ,[30] we

[28] It is argued in `HCSS-19121` that it might be preferable to follow the method that was used on-board and disregard the measurements from one of the four gyros.

[29] For the nominal (pre-launch) alignments, we have $G = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & -1 & 1\\ 1 & -1 & 1\\ 1 & -1 & -1\\ -1 & -1 & -1\end{pmatrix}$ and $G^{+} = \frac{\sqrt{3}}{4}\begin{pmatrix} -1 & 1 & 1 & -1\\ -1 & -1 & -1 & -1\\ 1 & 1 & -1 & -1\end{pmatrix}$ [1, pp. 599–600].

[30] Writing $\underline{\theta} = \theta\,\mathbf{e}$ and $\underline{\omega} = \begin{bmatrix}\omega_u & \omega_v & \omega_w\end{bmatrix}^{\mathrm{T}}$, (29) become

$$
\dot{\underline{\theta}} = \underline{\omega} + \frac{1}{2}\,\underline{\theta} \times \underline{\omega} + \mathrm{O}\,(\theta^2).
$$

obtain

$$\begin{bmatrix}\dot\theta_x\\ \dot\theta_y\\ \dot\theta_z\end{bmatrix} = G^+ \begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}. \tag{31}$$

Now, the Inertial Reference Unit (GYR) provides measurements of the integrated angular rotations, ϕ_1, ϕ_2, ϕ_3 and ϕ_4, of the spacecraft about each of the four gyro axes (from some unknown time). If we assume that their derivatives are related to the true rates about these axes by

$$\dot\phi_i = k_i\,\omega_i + b_i, \qquad i = 1, \dots, 4,$$

where k_i are the (known) scale factors and b_i are the (unknown) drift rates,[31] then (31) becomes

$$\begin{aligned}\dot\theta_x &= \dot\psi_x + b_x,\\ \dot\theta_y &= \dot\psi_y + b_y,\\ \dot\theta_z &= \dot\psi_z + b_z,\end{aligned} \tag{32}$$

where

$$\begin{bmatrix}\psi_x\\ \psi_y\\ \psi_z\end{bmatrix} \equiv G^+ \begin{bmatrix}\phi_1/k_1\\ \phi_2/k_2\\ \phi_3/k_3\\ \phi_4/k_4\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}b_x\\ b_y\\ b_z\end{bmatrix} = -G^+ \begin{bmatrix}b_1/k_1\\ b_2/k_2\\ b_3/k_3\\ b_4/k_4\end{bmatrix}. \tag{33}$$

Next, we assume that a value T be chosen such that the variation in the gyro drift rates (and hence in b_x, b_y and b_z) is negligible within the interval $I_k \equiv [t_k - T/2,\, t_k + T/2]$.[32] For $t \in I_k$, equations (32) can then be integrated

So neglecting the linear term in θ introduces an error in $\dot{\underline{\theta}}$ of magnitude $\dot\theta_{\rm err}$, where

$$\frac{\dot\theta_{\rm err}}{\dot\theta} \approx \frac{|\underline{\theta} \times \underline{\omega}|}{2\,|\underline{\omega}|} \le \frac{\theta}{2}.$$

For example, for $\theta = 1^\circ$, $\dot\theta_{\rm err}/\dot\theta < 0.01$.

[31] The function `calcGyroAttitude` reads the values of k_i from the ACMS product metadata.

[32] The appropriateness of using this 'constant drift rate' model remains to be investigated. (Alternative models, such as one in which the gyro drift rates vary linearly with the time, could be easily implemented.) The window length, T, is specified by the input parameter `wind_len`, the default value for which is currently 400 s; see [8, p. 7] and [22, p. 68].

to give:

$$\begin{aligned}\theta_x(t) &= \psi_x(t) + b_x\,(t - t_k) + c_x,\\ \theta_y(t) &= \psi_y(t) + b_y\,(t - t_k) + c_y,\\ \theta_z(t) &= \psi_z(t) + b_z\,(t - t_k) + c_z,\end{aligned} \tag{34}$$

where c_x, c_y, c_z are constants of integration.

Equations (34) constitute the model used to relate the small rotations, θ_x, θ_y and θ_z, from the reference attitude $\mathbf{q}_{0,k}$ to three angles, ψ_x, ψ_y and ψ_z, derived from the gyro output.

2.2.2 Estimation of the model parameters

To find (estimate) the parameters in (34) we make use of the improved attitude estimates constructed from the star tracker data (see Section 2.1). Measurements of ψ_x, ψ_y and ψ_z are combined with measurements of θ_x, θ_y and θ_z and a least-squares fit is performed.

Let t_b and t_e be the start and end times of the observation and suppose that within the interval $I_k \cap [t_\mathrm{b},\, t_\mathrm{e}]$ we have n_g measurements of the angles ψ_x, ψ_y and ψ_z at times $t_\mathrm{g}^{(1)}, \ldots, t_\mathrm{g}^{(n_\mathrm{g})}$, i.e.

$$\{(\psi_x^{(i)},\, \psi_y^{(i)},\, \psi_z^{(i)}) : i = 1, \ldots, n_\mathrm{g}\},$$

and n_s good quality measurements of the angles θ_x, θ_y and θ_z at times $t_\mathrm{s}^{(1)}$, $\ldots$, $t_\mathrm{s}^{(n_\mathrm{s})}$,[33] i.e.

$$\{(\theta_x^{(i)},\, \theta_y^{(i)},\, \theta_z^{(i)}) : i = 1, \ldots, n_\mathrm{s}\}.$$

Since $t_k \in I_k$, we may without loss of generality suppose that

$$t_k = t_\mathrm{g}^{(l)}, \tag{35}$$

[33] To allow the timing of the star tracker measurements to be synchronized with that of the gyro measurements, an offset set is first added, i.e. $t_\mathrm{s}^{(i)} = \tilde{t}_\mathrm{s}^{(i)} + \texttt{toff\_star}$, where $\tilde{t}_\mathrm{s}^{(i)}$ is the time of the measurement created by `calcStrAttitude` and `toff_star` is an input parameter which may be specified by the User. It is noted that when assigning the time to each attitude measurement the function `calcStrAttitude` subtracts 0.805 s from the record time associated with the particular set of star unit vectors. Therefore, the total offset is in fact `toff_star` − 0.805 s. The measurements $\theta_x^{(i)}$, $\theta_y^{(i)}$ and $\theta_z^{(i)}$ are considered to be of good quality whenever the p_taste associated with the improved attitude estimate $\widehat{\mathbf{q}}_\mathrm{str}^{(i)}$, see (24), is greater than the value of the input parameter `prob_thresh`.

for some $1 \leq l \leq n_\mathrm{g}$.

To obtain the measurements, $\theta_x^{(i)}$, $\theta_y^{(i)}$, $\theta_z^{(i)}$, from the reference attitude $\mathbf{q}_{0,k}$ and the improved attitude estimate $\widehat{\mathbf{q}}_\mathrm{str}^{(i)}$ we simply use (26) and (28), i.e.

$$\begin{bmatrix} \frac{\theta_x^{(i)}}{2} & \frac{\theta_y^{(i)}}{2} & \frac{\theta_z^{(i)}}{2} & 1 \end{bmatrix}^\mathrm{T} = \mathbf{q}_{0,k}^* \, \widehat{\mathbf{q}}_\mathrm{str}^{(i)}. \tag{36}$$

In doing this we are approximating $\sin \frac{\theta^{(i)}}{2}$ by $\frac{\theta^{(i)}}{2}$ (see p. 14) and so introducing errors of up to $2 \left| \sin \frac{\theta^{(i)}}{2} - \frac{\theta^{(i)}}{2} \right| = \frac{(\theta^{(i)})^3}{24} + \mathrm{O}\left([\theta^{(i)}]^5\right)$, i.e. approximately $0.006''$ ($\theta^{(i)} = 0.5°$), $0.05''$ ($\theta^{(i)} = 1°$) and $0.4''$ ($\theta^{(i)} = 2°$). To ensure that these systematic errors do not corrupt the results, any measurement for which the rotation, $\theta^{(i)}$, from the reference attitude is greater than the parameter `rot_limit` is not used in the fitting.[34]

Typically there are four times as many measurements of ψ_x, ψ_y and ψ_z as of θ_x, θ_y and θ_z, i.e. $n_\mathrm{g} \approx 4\, n_\mathrm{s}$, and the times $\{t_\mathrm{g}^{(i)} : i = 1, \ldots, n_\mathrm{g}\}$ and $\{t_\mathrm{s}^{(i)} : i = 1, \ldots, n_\mathrm{s}\}$ do not 'coincide'. Moreover, the noise on the star tracker measurements, and hence on $\theta_x^{(i)}$, $\theta_y^{(i)}$ and $\theta_z^{(i)}$, is far greater than that arising from the gyro measurements. It follows that, to combine the two sets of measurements, it is appropriate to interpolate the measurements $\psi_x^{(i)}$, $\psi_y^{(i)}$, $\psi_z^{(i)}$ to the times $\{t_\mathrm{s}^{(i)} : i = 1, \ldots, n_\mathrm{s}\}$ of the star tracker measurements. For this, the software uses linear interpolation, i.e.

$$\widetilde{\psi}_x^{(i)} = \frac{\psi_x^{(j+1)}\,[t_\mathrm{s}^{(i)} - t_\mathrm{g}^{(j)}] + \psi_x^{(j)}\,[t_\mathrm{g}^{(j+1)} - t_\mathrm{s}^{(i)}]}{t_\mathrm{g}^{(j+1)} - t_\mathrm{g}^{(j)}}, \qquad t_\mathrm{g}^{(j)} \leq t_\mathrm{s}^{(i)} < t_\mathrm{g}^{(j+1)},$$

and similarly for $\widetilde{\psi}_y^{(i)}$ and $\widetilde{\psi}_z^{(i)}$.

Using this set of $3\, n_\mathrm{s}$ measurements with the model (34), we obtain the

[34] This is achieved by assigning the measurement a zero weight when solving (39) and incrementing the number, n_d, of discarded measurements.

following linear regression equations:

$$\begin{bmatrix} \theta_x^{(1)} - \widetilde{\psi}_x^{(1)} \\ \theta_y^{(1)} - \widetilde{\psi}_y^{(1)} \\ \theta_z^{(1)} - \widetilde{\psi}_z^{(1)} \\ \cdots \\ \theta_x^{(n_s)} - \widetilde{\psi}_x^{(n_s)} \\ \theta_y^{(n_s)} - \widetilde{\psi}_y^{(n_s)} \\ \theta_z^{(n_s)} - \widetilde{\psi}_z^{(n_s)} \end{bmatrix} = \begin{bmatrix} t_\mathrm{s}^{(1)} - t_k & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_\mathrm{s}^{(1)} - t_k & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_\mathrm{s}^{(1)} - t_k & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_\mathrm{s}^{(n_s)} - t_k & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & t_\mathrm{s}^{(n_s)} - t_k & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & t_\mathrm{s}^{(n_s)} - t_k & 1 \end{bmatrix} \begin{bmatrix} b_x \\ c_x \\ b_y \\ c_y \\ b_z \\ c_z \end{bmatrix} + \begin{bmatrix} \epsilon_x^{(1)} \\ \epsilon_y^{(1)} \\ \epsilon_z^{(1)} \\ \cdots \\ \epsilon_x^{(n_s)} \\ \epsilon_y^{(n_s)} \\ \epsilon_z^{(n_s)} \end{bmatrix},$$

which we write, more concisely, as

$$y = X\,\beta + \epsilon. \tag{37}$$

The error term, ϵ, is assumed to have a (conditional) mean of zero and (conditional) variance given by the block diagonal matrix:

$$\Omega = \mathrm{diag}(P_{\theta\theta}^{(1)}, \ldots, P_{\theta\theta}^{(n_s)}),$$

where $P_{\theta\theta}^{(i)}$ is the Cartesian attitude covariance matrix of measurement i, computed according to (25).

The best linear unbiased estimate of β (i.e. of the parameters b_x, c_x, …, b_z, c_z) may be obtained by using the generalized least-squares method to minimize $\epsilon^\mathrm{T}\,\Omega^{-1}\,\epsilon$, i.e. by solving the normal equations

$$(X^\mathrm{T}\,\Omega^{-1}\,X)\,\hat{\beta} = X^\mathrm{T}\,\Omega^{-1}\,y. \tag{38}$$

The software assumes that the off-diagonal elements of $P_{\theta\theta}^{(i)}$ are negligible.[35] The problem then becomes one which may be solved by a weighted least-squares method and (38) decouple to give:

$$\begin{aligned} (X_x^\mathrm{T}\,X_x) \begin{bmatrix} \hat{b}_x \\ \hat{c}_x \end{bmatrix} &= X_x^\mathrm{T}\,y_x, \\ (X_y^\mathrm{T}\,X_y) \begin{bmatrix} \hat{b}_y \\ \hat{c}_y \end{bmatrix} &= X_y^\mathrm{T}\,y_y, \\ (X_z^\mathrm{T}\,X_z) \begin{bmatrix} \hat{b}_z \\ \hat{c}_z \end{bmatrix} &= X_z^\mathrm{T}\,y_z, \end{aligned} \tag{39}$$

[35] However, this has been found to be far from true. As might be expected from geometrical considerations, the errors about the spacecraft x-axis are highly correlated with those about the y- and z-axes.

where

$$X_x = \begin{bmatrix} (t_\text{s}^{(1)} - t_k)/\sigma_x^{(1)} & 1/\sigma_x^{(1)} \\ (t_\text{s}^{(2)} - t_k)/\sigma_x^{(2)} & 1/\sigma_x^{(2)} \\ \cdots & \cdots \\ (t_\text{s}^{(n_\text{s})} - t_k)/\sigma_x^{(n_\text{s})} & 1/\sigma_x^{(n_\text{s})} \end{bmatrix}, \quad y_x = \begin{bmatrix} (\theta_x^{(1)} - \widetilde{\psi}_x^{(1)})/\sigma_x^{(1)} \\ (\theta_x^{(2)} - \widetilde{\psi}_x^{(2)})/\sigma_x^{(2)} \\ \cdots \\ (\theta_x^{(n_\text{s})} - \widetilde{\psi}_x^{(n_\text{s})})/\sigma_x^{(n_\text{s})} \end{bmatrix},$$

$\sigma_x^{(i)} = \sqrt{P_{\theta\theta}^{(i)}[1,1]}$ and similary for X_y, y_y, σ_y and X_z, y_z, σ_z.[36]

Solving (39) is equivalent to minimizing the cost functions:

$$\chi_r^2(b_r, c_r) = \left(y_r - X_r \begin{bmatrix} b_r \\ c_r \end{bmatrix}\right)^\text{T} \left(y_r - X_r \begin{bmatrix} b_r \\ c_r \end{bmatrix}\right), \quad r \in \{x, y, z\}.$$

Assuming each of the (normalized) errors contributing to the sum $\chi_r^2(b_r, c_r)$ to be normally distributed, we expect $\chi_r^2(\hat{b}_r, \hat{c}_r)$ to come from a chi-squared distribution with $n_\text{s} - n_\text{d} - 2$ degrees of freedom [14, p. 654], where n_d is the number of discarded measurements (see footnote 34). As a measure of the goodness-of-fit we may therefore use the probability of exceeding the value $\chi_r^2(\hat{b}_r, \hat{c}_r)$ at random, viz.

$$\Pr\{\min\{\chi_r^2\} > \chi_r^2(\hat{b}_r, \hat{c}_r)\} = 1 - P\left(\frac{n_\text{s} - n_\text{d} - 2}{2}, \frac{\chi_r^2(\hat{b}_r, \hat{c}_r)}{2}\right), \tag{40}$$

where $r \in \{x, y, z\}$ and P is the lower (regularized) incomplete gamma function.

2.2.3 Computation of the new attitude estimate and its uncertainty

Assuming the fits are good, (34) and (35) may be used to obtain estimates of the small-angle rotations at time t_k, i.e.

$$\widehat{\theta}_r(t_k) = \psi_r^{(l)} + \hat{c}_r,, \quad r \in \{x, y, z\}. \tag{41}$$

Then from (28),

$$\widehat{\mathbf{q}}_r^{(k)}(t_k) = \begin{bmatrix} \frac{\widehat{\theta}_x(t_k)}{2} & \frac{\widehat{\theta}_y(t_k)}{2} & \frac{\widehat{\theta}_z(t_k)}{2} & 1 \end{bmatrix}^\text{T}, \tag{42}$$

[36] $P_{\theta\theta}^{(i)}[j,\, k]$ indicates the jk^th component of $P_{\theta\theta}^{(i)}$.

and, finally, from (26),

$$\widehat{\mathbf{q}}_{\mathrm{aca}}(t_k) = \mathbf{q}_{0,k}\, \widehat{\mathbf{q}}_r^{(k)}(t_k). \tag{43}$$

Let $\tilde{\theta}_{x,k}$, $\tilde{\theta}_{y,k}$ and $\tilde{\theta}_{z,k}$ be the errors in the small angle rotations at time t_k. In passing from (38) to (39) we are ignoring the coupling between these errors and so we are unable to compute $\mathrm{E}[\tilde{\theta}_{x,k}\, \tilde{\theta}_{y,k}]$, $\mathrm{E}[\tilde{\theta}_{x,k}\, \tilde{\theta}_{z,k}]$ and $\mathrm{E}[\tilde{\theta}_{y,k}\, \tilde{\theta}_{z,k}]$ (see description of `HCSS-19472` on p. 43). However, the diagonal elements of the Cartesian attitude covariance matrix contain the expected variances of the errors in c_x, c_y and c_z and these are readily obtained, i.e.

$$\mathrm{E}[\tilde{\theta}_{r,k}^2] = \mathrm{E}(\tilde{c}_r^2) = (X_r^{\mathrm{T}}\, X_r)^{-1}[2,2], \quad r \in \{x, y, z\}. \tag{44}$$

3 How to use the software

To obtain a pointing product containing the attitudes reconstructed by the gyro-based method, use `calcAttitude`. This task may be either selected and run from the task view or called from the command line:

newpp = calcAttitude (oldpp, acmsProduct, tcHistoryProduct,
[prob_thresh = ..., ref_thresh = ...,
wind_len = ..., rot_limit = ..., toff_star = ...,
excl_gyro = ...])

By calling first `calcStrAttitude` and then `calcGyroAttitude`, the task `calcAttitude` takes the information contained in the ACMS telemetry product, `acmsProduct`, and uses the method described in Section 2 to estimate the ACA-frame attitude at a sequence of times spanning the period of the original pointing product, `oldpp`.[37] It then creates a new copy, `newpp`, of the pointing product and augments this with the newly estimated attitudes and the associated quality and accuracy information. To do this, the task loops through the lines of the pointing product and replaces each of the attitude quaternions in the column labeled `filterQuat` with the newly-estimated attitude, $\widehat{\mathbf{q}}_{\rm aca}(t_k)$, associated with the closest matching time, t_k. The related quality and accuracy information are written to the columns labelled `gyroAttProbX`, `gyroAttProbY`, `gyroAttProbZ` and `gyroAttSigmaX`, `gyroAttSigmaY`, `gyroAttSigmaZ` respectively. To indicate that this has been performed, the flag in the column labelled `filterQuatFlag` is set equal to 0. If, for some time t_k, the gyro-based method was unable to estimate the attitude, then the quaternion in the column labeled `filterQuat` and the flag in the column labelled `filterQuatFlag` are left unchanged. Irrespective of whether the values in column `filterQuat` are overwritten, the old attitude quaternion—which is that computed by the on-board filter (with or without the simple focal length correction)—is copied to the column labelled `simpleCorrFilterQuat`.

A summary of the columns in the pointing product associated with the gyro-based attitude reconstruction is given below in Table 1; for a complete

[37] As explained in Section 2.2, the gyro-based method reconstructs the attitudes at the times associated with the gyro measurements. Although these times are very similar to the OBT times in the pointing product, they do not match exactly (see also Section 4, `HCSS-19201`). The TC history product, `tcHistoryProduct`, is also required as it contains the spacecraft velocity vector which is needed in the aberration correction.

description of all the columns, see [16]. In addtion, `calcStrAttitude` accepts a number of optional input parameters; these are described in Table 2.

Column	Field	Units	Description
1	obt	μs	On-board time.
3	filterQuat	None	Reconstructed ACA-frame attitude quaternion. Iff filterQuatFlag = 0, then filterQuat contains $\widehat{\mathbf{q}}_{\text{aca}}(t_k)$, where t_k is the time of the closest gyro measurement.
19	gyroAttProbX	None	Quality associated with x-axis fit, i.e. $\Pr\{\min\{\chi_x^2\} > \chi_x^2(\hat{b}_x, \hat{c}_x)\}$.[38]
20	gyroAttProbY	None	Quality associated with y-axis fit, i.e. $\Pr\{\min\{\chi_y^2\} > \chi_y^2(\hat{b}_y, \hat{c}_y)\}$.
21	gyroAttProbZ	None	Quality associated with z-axis fit, i.e. $\Pr\{\min\{\chi_z^2\} > \chi_z^2(\hat{b}_z, \hat{c}_z)\}$.
22	gyroAttSigmaX	arcsec.	Standard deviation of error in gyro-based reconstructed attitude about ACA-frame x-axis, i.e. $\sqrt{\mathrm{E}[\tilde{\theta}_{x,k}^2]}$.[39]
23	gyroAttSigmaY	arcsec.	Standard deviation of error in gyro-based reconstructed attitude about ACA-frame y-axis, i.e. $\sqrt{\mathrm{E}[\tilde{\theta}_{y,k}^2]}$.
24	gyroAttSigmaZ	arcsec.	Standard deviation of error in gyro-based reconstructed attitude about ACA-frame z-axis, i.e. $\sqrt{\mathrm{E}[\tilde{\theta}_{z,k}^2]}$.
25	filterQuatFlag	None	Flag indicating nature of attitude in filterQuat field. (0 = gyro-based method, 1 = simple focal length correction, 2 = on-board filter)

Table 1: Relevant columns of pointing product

[38] See p. 20, eq. (40).
[39] See p. 21, eq. (44).

Before using the attitudes computed by the gyro-based method, it is important that the User first check the quality of the results. For example, a plot for observation 1342197884 of the probabilities found in columns 19–21 (Figure 1) indicates that the method was successful and that the reconstructed attitudes, as shown by the blue curves in Figures 2–4, are likely to be of good quality.[40] The 1-σ uncertainties about the spacecraft axes are shown in Figure 5.

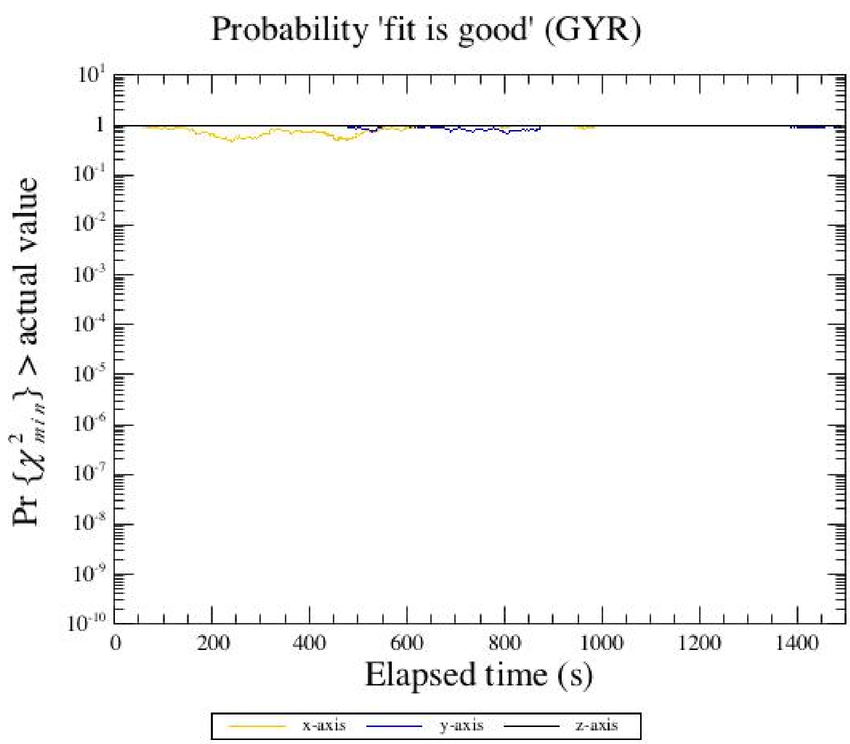

Figure 1: Quality plot – 1342197884

[40]For this particular observation—in fact for much of OD 389—the attitudes from the gyro-based method are in closer agreement with those from the on-board filter than with the 'simple-corrected' attitudes. It is known that the 'simple-corrected' attitudes are likely to be inaccurate whenever interlacing mode was active; see `HCSS-19870`.

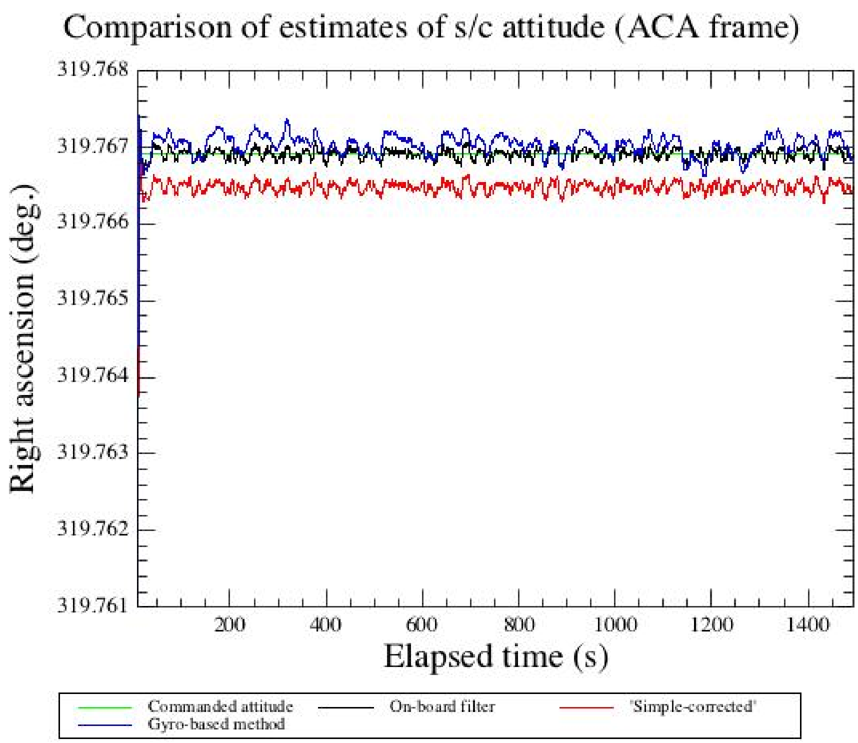

Figure 2: Right ascension – 1342197884

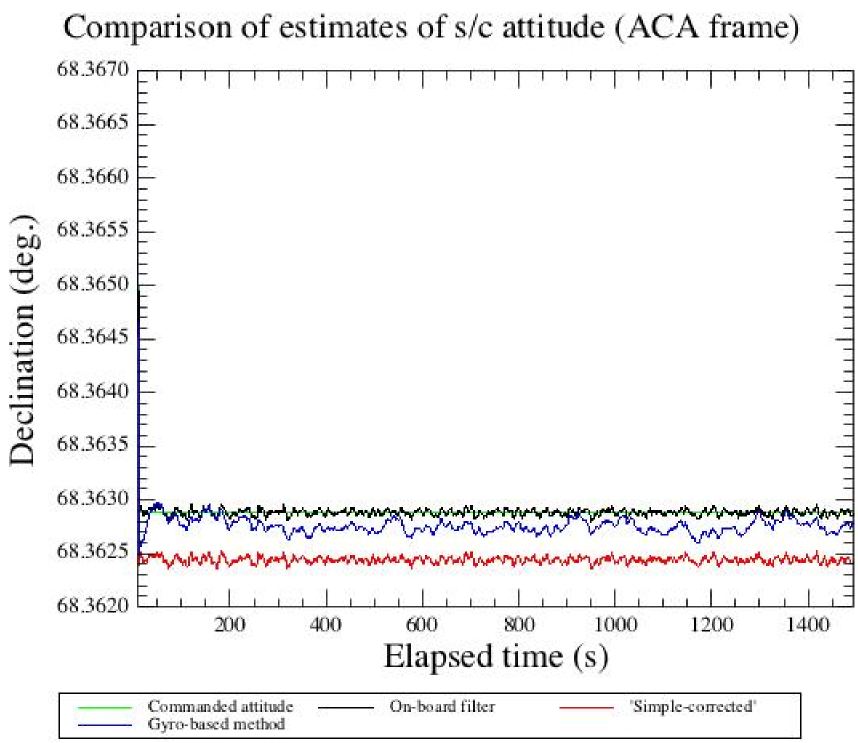

Figure 3: Declination – 1342197884

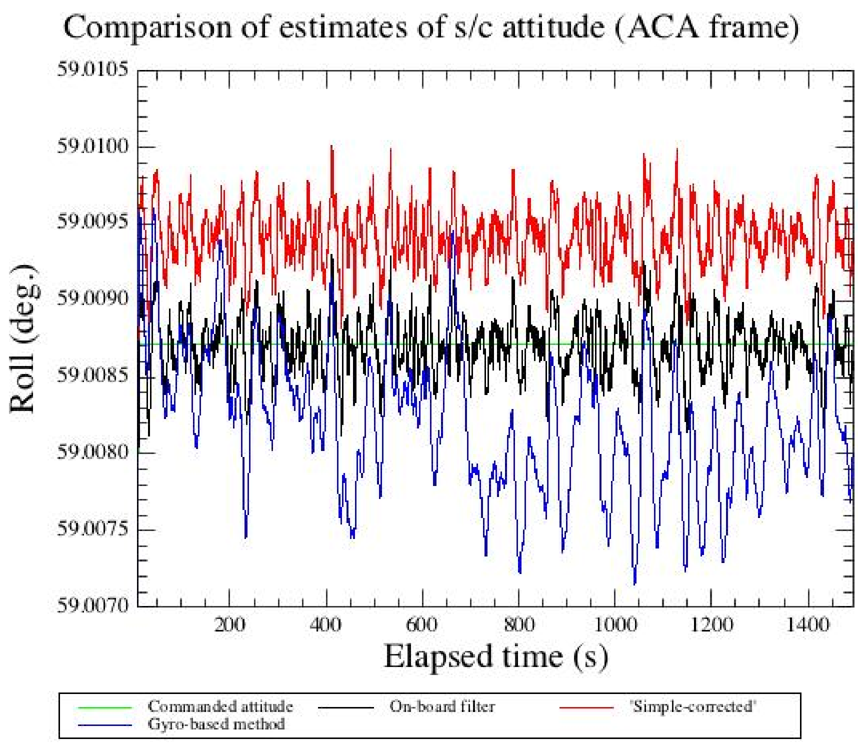

Figure 4: Roll – 1342197884

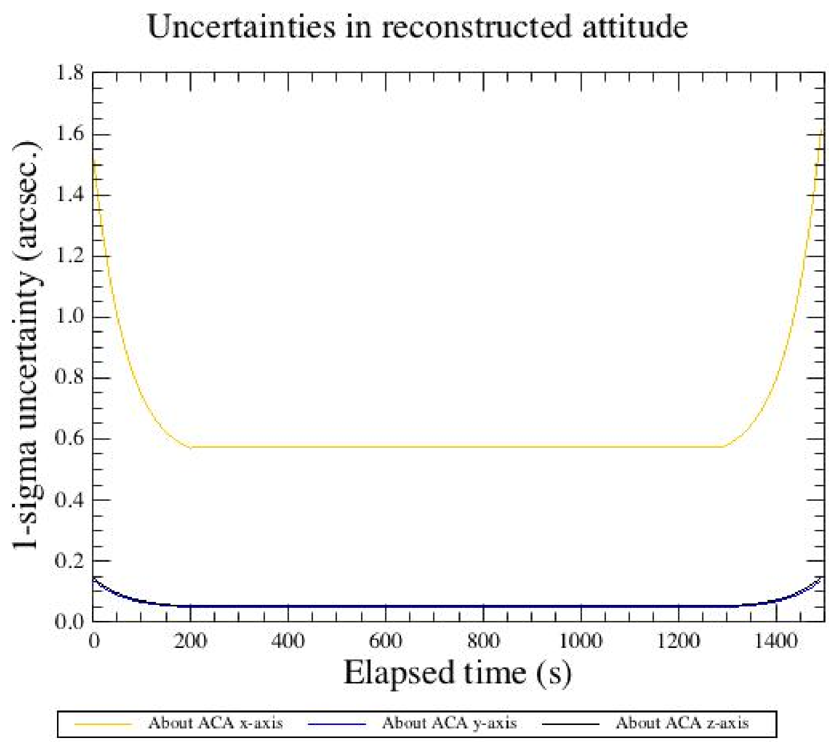

Figure 5: 1-σ uncertainties – 1342197884

On the other hand, for 1342227764 the quality plot (Figure 6) shows that, for most of the duration of this observation, little confidence can be placed in the reconstructed attitude. The associated uncertainties for this observation are meaningless and so are not shown.

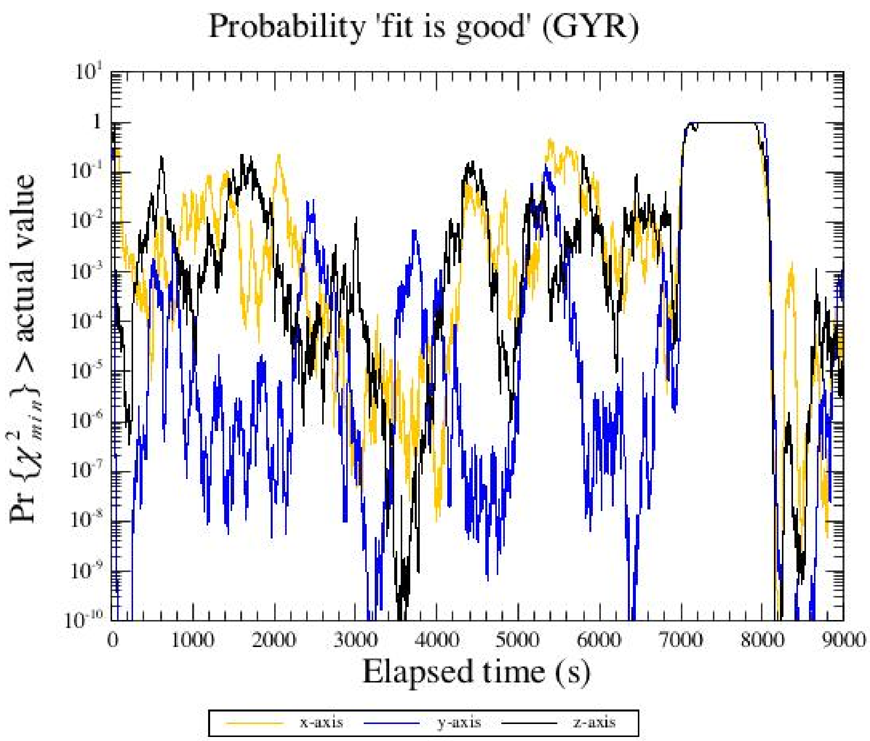

Figure 6: Quality plot – 1342227764

To investigate cases like this further, additional output from the attitude reconstruction may be useful. Such output may be produced by calling the functions `calcStrAttitude` and `calcGyroAttitude` directly and setting the debug flags, i.e.

(strAttitude, status) = calcStrAttitude (oldpp, acmsProduct,
tcHistoryProduct,
[prob_thresh = ..., debug = 1])

if (status == 0):

gyroAttitude = calcGyroAttitude (oldpp, acmsProduct, strAttitude,
[ref_thresh = . . . , wind_len = . . . ,
rot_limit = . . . , toff_star = . . . ,
prob_thresh = . . . , excl_gyro = . . . ,
debug = 1])

The first call produces a table, `strAttitude`, containing the corrected attitude measurements made by the (operational) star tracker. These are then passed to the function `calcGyroAttitude`, which produces a table `gyroAttitude` containing the reconstructed attitude. Again, a full description of the various input and output parameters for these two calls is given in Table 2.[41].

The quality of the attitude measurements produced by `calcStrAttitude` may be examined by means of the probability found in column 11 (`prob_taste`) of `strAttitude` (see Table 3). Figures 7 and 8 show this probability for the two observations considered above. It is clear from these figures that there is a problem with the construction of the attitude measurements for observation 1342227764.

When the debug flag is set, `calcStrAttitude` also estimates the attitude using the uncorrected star vectors. The quality of this estimation is indicated by the probability contained in column 38 (`prob_taste_old`). Comparing, for observation 1342197884, a plot of this probability (Figure 9) with that shown in Figure 7, we get an idea of the improvement in accuracy which has been achieved through the application of the new distortion correction.[42]

[41] Unlike `calcAttitude`, this does not produce a new pointing product, but it has the advantage that the time stamps of the quaternions are slightly more accurate (see Section 4, `HCSS-19201`)

[42] Since the software currently uses a maximum of 9 stars, this gain in accuracy will be offset to some extent whenever interlacing mode is active.

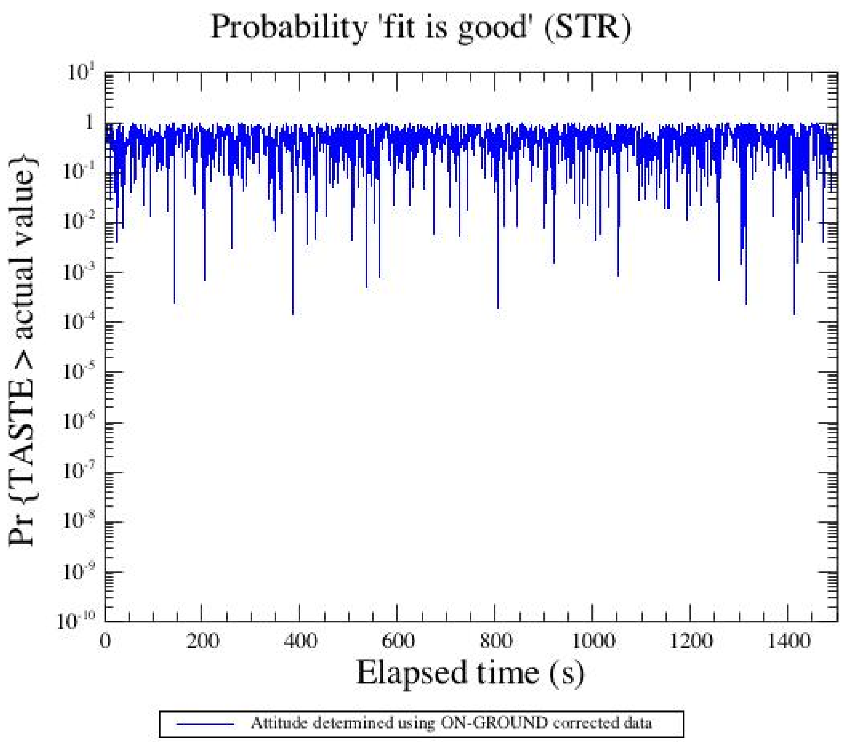

Figure 7: Quality of attitude measurements – 1342197884

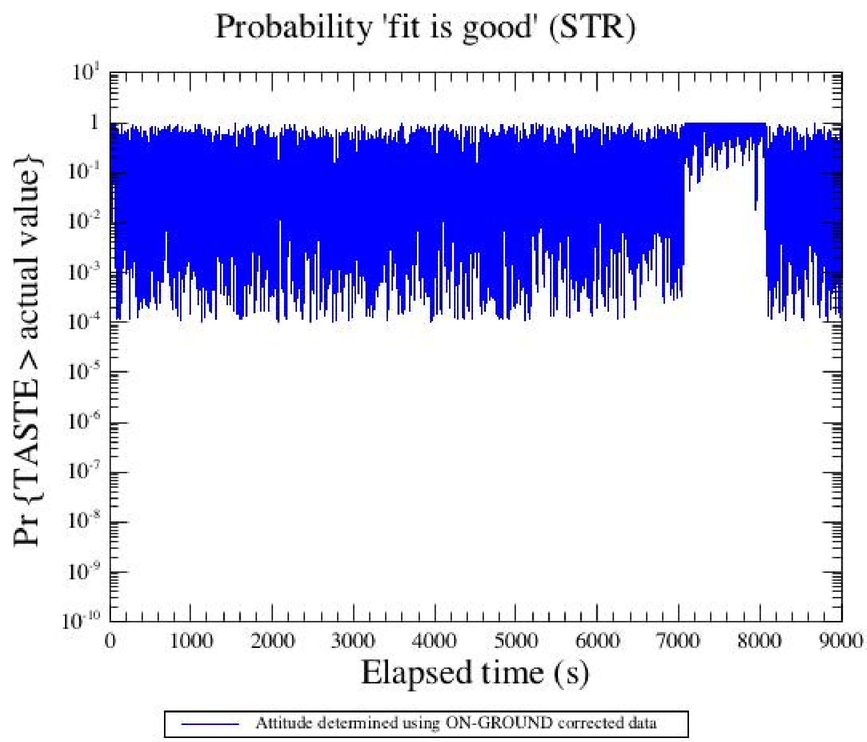

Figure 8: Quality of attitude measurements – 1342227764

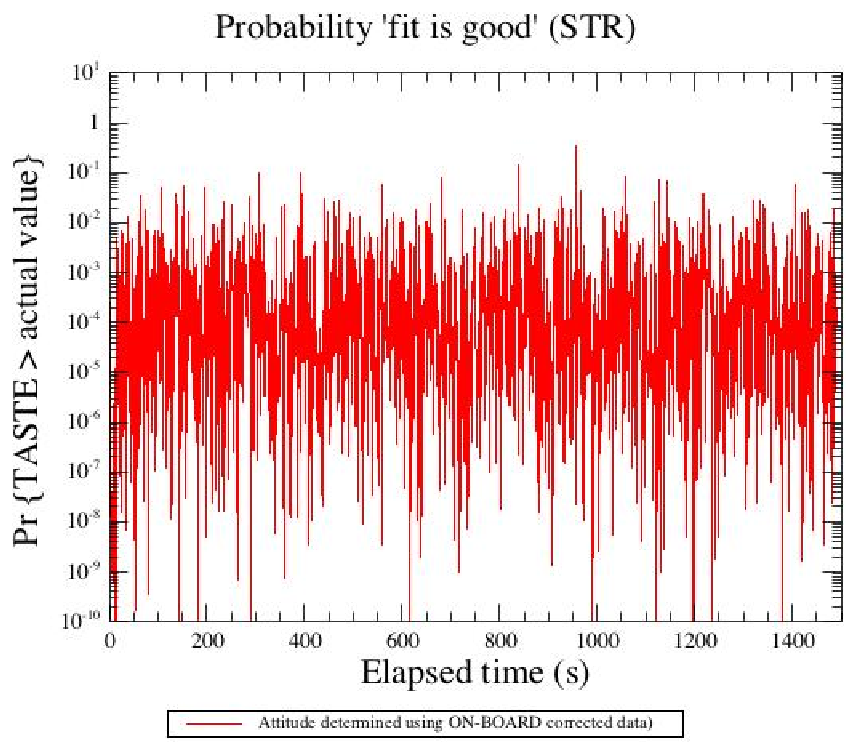

Figure 9: Quality of uncorrected attitude measurements – 1342197884

Parameter	Class	Description
oldpp	PointingProduct	Original pointing product, i.e. `obs.auxiliary.pointing` where `obs` is an observation context.
acmsProduct	AcmsTelemetryProduct	ACMS telemetry product, i.e. `obs.auxiliary.acms` where `obs` is an observation context.
tcHistoryProduct	TeleCommandHistProduct	TC history product, i.e. `obs.auxiliary.teleCommHistory` where `obs` is an observation context.
newpp	PointingProduct	New pointing product.
prob_thresh	Double	Probability threshold used for deciding when fit to determine star tracker attitude is bad (optional, default value $= 1.0 \times 10^{-4}$); see p. 13.
ref_thresh	Double	The reference attitude is updated whenever it is found to differ by more than ref_thresh (arcsecs.) from the 'current' star tracker attitude measurement (optional, default value = 100.0); see p. 14, footnote 27.
wind_len	Double	The nominal length (in seconds) of the of the interval used to estimate the parameters in the least-squares fits (optional, default value = 400.0); see p. 16, footnote 32.
rot_limit	Double	Attitude measurements which differ by more than rot_limit (deg.) from the reference attitude are excluded from the fitting (optional, default value = 0.5); see p. 18.
toff_star	Double	Time offset (in seconds) used to synchronize the star tracker attitude measurements with the gyro measurements (optional, default value

		= 0.189); see p. 17, footnote 33.[43]
excl_gyro	Int	Number of the gyro from which measurements are to be excluded. If set equal to zero, or any integer other than {1,2,3,4}, then measurements from all four gyros are used (optional, default value = 0).
debug	Int	Set equal to 1 for additional output (optional, default value = 0).
strAttitude	TableDataset	Contains corrected attitude measurements (from star tracker); see Table 3.
status	Int	Return status from calcStrAttitude (0 = success, otherwise error). See function header for further details.
gyroAttitude	TableDataset	Contains reconstructed attitude; see Table 4.

Table 2: Input and output parameters

[43] The default value for `toff_star` is based upon the value which was estimated in [11] for being optimal for the now-defunct parameter `toff_gyro`; see `HCSS-19454` for further details.

Column	Name	Type	Units	Description
1	obt	Double1d(*)	μs	On-board time.
2	newmeas_q	Double2d(*, 4)	None	New (corrected) measurement of ACA-frame attitude quaternion.
3	newmeas_ra	Double1d(*)	deg.	New (corrected) measurement of right ascension of ACA-frame x-axis.
4	newmeas_dec	Double1d(*)	deg.	New (corrected) measurement of declination of ACA-frame x-axis.
5	newmeas_roll	Double1d(*)	deg.	New (corrected) measurement of roll about ACA-frame x-axis.
6	newstr_q	Double2d(*, 4)	None	New (corrected) measurement of STR-frame attitude quaternion.
7	newstr_ra	Double1d(*)	deg.	New (corrected) measurement of right ascension of STR-frame x-axis.
8	newstr_dec	Double1d(*)	deg.	New (corrected) measurement of declination of STR-frame x-axis.
9	newstr_roll	Double1d(*)	deg.	New (corrected) measurement of roll about STR-frame x-axis.
10	TASTE	Double1d(*)	None	Value of TASTE variable from attitude determination.
11	prob_taste	Double1d(*)	None	Probability, p_{taste}, of such a large value of TASTE occurring at random; see p. 13, eq. (24).
12	num_bad	Int1d(*)	None	Number of stars excluded from attitude determination.
13	badStars	Int2d(*, 2)	None	IDs of excluded stars (if any).
14	nStarsUsed	Int1d(*)	None	Number of stars used in attitude determination.
15	sigma_x	Double1d(*)	arcsec.	Standard deviation of error about ACA-frame x-axis.
16	sigma_y	Double1d(*)	arcsec.	Standard deviation of error about ACA-frame y-axis.
17	sigma_z	Double1d(*)	arcsec.	Standard deviation of error about ACA-frame z-axis.
18	rho_yz	Double1d(*)	None	Correlation coefficient between errors about ACA-frame y- and z-axes.

19	rho_xz	Double1d(*)	None	Correlation coefficient between errors about ACA-frame x- and z-axes.
20	rho_xy	Double1d(*)	None	Correlation coefficient between errors about ACA-frame x- and y-axes.
21 (debug)	obmeas_q	Double2d(*, 4)	None	On-board measurement of ACA-frame attitude quaternion.
22 (debug)	obmeas_ra	Double1d(*)	deg.	On-board measurement of right ascension of ACA-frame x-axis.
23 (debug)	obmeas_dec	Double1d(*)	deg.	On-board measurement of declination of ACA-frame x-axis.
24 (debug)	obmeas_roll	Double1d(*)	deg.	On-board measurement of roll about ACA-frame x-axis.
25 (debug)	oldmeas_q	Double2d(*, 4)	None	Uncorrected measurement of ACA-frame attitude quaternion.[44]
26 (debug)	oldmeas_ra	Double1d(*)	deg.	Uncorrected measurement of right ascension of ACA-frame x-axis.
27 (debug)	oldmeas_dec	Double1d(*)	deg.	Uncorrected measurement of declination of ACA-frame x-axis.
28 (debug)	oldmeas_roll	Double1d(*)	deg.	Uncorrected measurement of roll about ACA-frame x-axis.
29 (debug)	rot_ob_q	Double2d(*, 4)	None	Quaternion giving rotation of new attitude measurement with respect to on-board attitude measurement.
30 (debug)	rot_ob_x	Double1d(*)	arcsec.	ACA-frame x-axis component of new attitude measurement with respect to on-board attitude measurement.
31 (debug)	rot_ob_y	Double1d(*)	arcsec.	ACA-frame y-axis component of new attitude measurement with respect to on-board attitude measurement.
32 (debug)	rot_ob_z	Double1d(*)	arcsec.	ACA-frame z-axis component of new attitude measurement with respect to on-board attitude measurement.
33 (debug)	rot_old_q	Double2d(*, 4)	None	Quaternion giving rotation of new attitude measurement with respect to uncorrected attitude measurement.

[44] 'Uncorrected' means computed on-ground using the measured star vectors in (6).

34 (debug)	rot_old_x	Double1d(*)	arcsec.	ACA-frame x-axis component of new attitude measurement with respect to uncorrected attitude measurement.
35 (debug)	rot_old_y	Double1d(*)	arcsec.	ACA-frame y-axis component of new attitude measurement with respect to uncorrected attitude measurement.
36 (debug)	rot_old_z	Double1d(*)	arcsec.	ACA-frame z-axis component of new attitude measurement with respect to uncorrected attitude measurement.
37 (debug)	TASTE_old	Double1d(*)	None	Value of TASTE variable from attitude determination giving oldmeas_q.
38 (debug)	prob_taste_old	Double1d(*)	None	Probability, p_{taste}, of such a large value of TASTE (that in TASTE_old) occurring at random; see p. 13, eq. (24).

Table 3: Contents of strAttitude

Column	Name	Type	Units	Description
1	obt	Double1d(*)	μs	On-board time, t_k, of gyro measurement.
2	gyratt_q	Double2d(*, 4)	None	Reconstructed ACA-frame attitude quaternion, $\widehat{\mathbf{q}}_{\text{aca}}(t_k)$.
3	gyratt_ra	Double1d(*)	deg.	Reconstructed right ascension of ACA-frame x-axis.
4	gyratt_dec	Double1d(*)	deg.	Reconstructed declination of ACA-frame x-axis.
5	gyratt_roll	Double1d(*)	deg.	Reconstructed roll about ACA-frame x-axis.
6	prob_x	Double1d(*)	None	Quality associated with x-axis fit, i.e. $\Pr\{\min\{\chi_x^2\} > \chi_x^2(\hat{b}_x, \hat{c}_x)\}$, see eq. (40), p. 20.
7	prob_y	Double1d(*)	None	Quality associated with y-axis fit, i.e. $\Pr\{\min\{\chi_y^2\} > \chi_y^2(\hat{b}_y, \hat{c}_y)\}$, see eq. (40), p. 20.
8	prob_z	Double1d(*)	None	Quality associated with z-axis fit, i.e. $\Pr\{\min\{\chi_z^2\} > \chi_z^2(\hat{b}_z, \hat{c}_z)\}$, see eq. (40), p. 20.
9	sigma_x	Double1d(*)	arcsec.	Standard deviation of error in gyro-based reconstructed attitude about ACA-frame x-axis, i.e. $\sqrt{\mathrm{E}[\tilde{\theta}_{x,k}^2]}$, see eq. (44), p. 21.
10	sigma_y	Double1d(*)	arcsec.	Standard deviation of error in gyro-based reconstructed attitude about ACA-frame y-axis, i.e. $\sqrt{\mathrm{E}[\tilde{\theta}_{y,k}^2]}$, see eq. (44), p. 21.
11	sigma_z	Double1d(*)	arcsec.	Standard deviation of error in gyro-based reconstructed attitude about ACA-frame z-axis, i.e. $\sqrt{\mathrm{E}[\tilde{\theta}_{z,k}^2]}$, see eq. (44), p. 21.
12 (`debug`)	maxrot_ref	Double1d(*)	deg.	Maximum rotation—over the interval used for the estimation—of the star tracker measurements

				from the reference attitude.
13 (debug)	num_meas	Int1d(*)	None	Number of measurements used in the fit (for each axis).
14 (debug)	drift_x	Double1d(*)	arcsec./s	Estimated (gyro) drift rate about ACA-frame x-axis.
15 (debug)	drift_y	Double1d(*)	arcsec./s	Estimated (gyro) drift rate about ACA-frame y-axis.
16 (debug)	drift_z	Double1d(*)	arcsec./s	Estimated (gyro) drift rate about ACA-frame z-axis.
17 (debug)	sig_dr_x	Double1d(*)	arcsec./s	Standard deviation of error in estimated (gyro) drift rate about ACA-frame x-axis.
18 (debug)	sig_dr_y	Double1d(*)	arcsec./s	Standard deviation of error in estimated (gyro) drift rate about ACA-frame y-axis.
19 (debug)	sig_dr_z	Double1d(*)	arcsec./s	Standard deviation of error in estimated (gyro) drift rate about ACA-frame z-axis.

Table 4: Contents of gyroAttitude

4 Summary of known issues

Below is a summary of the issues known to affect the versions of the functions contained in build 4772 of HIPE 13.0 or build 214 of HIPE 14.0: `CalcAttitudeTask` (version 1.6), `calcStrAttitude` (version 1.30) and `calcGyroAttitude` (version 1.29).[45] For further details, click on the 'issue key' to follow the link to the JIRA page describing the issue.

Table 5 provides an overview of the issues and attempts to classify them according to the likelihood of the User encountering the problem and the severity of the problem if encountered.

Issue key HCSS-...	**Likelihood of problem** Unlikely (0), Probable (1), Very likely (2), Always (3)	**Severity** No impact (0), Minor (1), Moderate (2), Major (3)
18766	1–2	**3**
19121	0–1	2
19122	1	2
19200	3	0
19201	0–1	**3**
19219	0–1	**3**
19267	3	0–1
19293	3	0
19463	1	**3**
19472	3	1(?)
19775	0	2

Table 5: Overview of issues

`HCSS-18766`

Large ($\sim 2''$ amplitude) oscillations have been observed in the residuals from the simple linear regressions (least-squares fits) which the function `calcGyroAttitude` performs.

`HCSS-19121`

The Moore–Penrose pseudoinverse, G^+, of the gyro alignment matrix

[45] The exception to this is `HCSS-19122`, which is associated with the software used to generate the star tracker CCD distortion maps. Since the function `calcAttitude` calls both `calcStrAttitude` and `calcGyroAttitude`, all issues affecting the latter two functions also affect `calcAttitude`.

(see p. 15) is being used in (33) to perform a least-squares fit and convert the four integrated gyro rates into three small-angle rotations. The function `calcGyroAttitude` currently does not check to see whether the output from the four gyros is consistent.

`HCSS-19122`
The software used to compute the star tracker CCD distortion maps compares the measured coordinates of each detected star with its expected coordinates. However, instead of using the best estimate of the spacecraft attitude to transform the catalogue coordinates of each star from the inertial reference frame to the Boresight Reference Frame (BRF), this software currently uses the attitude estimated using the 'reference period' parameters (i.e. the parameters which were on-board between ODs 320–762). New distortion maps have now been generated [see 9], but these are still not being used!

`HCSS-19200`
The creation date added by `calcAttitude` to the meta data of the new pointing product is not the date when the product was created.

`HCSS-19201`
The function `calcAttitude` overwrites the fields `filterQuat`, which contain the attitude quaternions from the on-board filter, with the attitude quaternions from the ground-based attitude reconstruction. Since the OBTs in the pointing product are left unmodified and since they do not correspond exactly with those found in the output table from `calcGyroAttitude`, each quaternion is overwritten with the quaternion corresponding to the closest matching time. Typically, the times match to within a few microseconds, but, as no test is performed, the mismatch may be larger. (Times differing by as much as 1.5 seconds have been observed.)

`HCSS-19219`
The gyro-based software will produce completely erroneous results whenever STR-B was the operational star tracker (mainly due to the use of the incorrect star tracker alignment quaternion). In the auxiliary processors this problem has been mitigated by only overwriting the `filterQuat` attitudes in the pointing product for periods when STR-A

was operational. In addition, the software currently cannot handle spacecraft velocity vector resets.

`HCSS-19267`
: The standard deviation of the error used in the QUEST measurement model is currently hard-coded to a value which was estimated, using (23), for just a handful of observations. It would be a good idea to re-estimate this parameter within `calcStrAttitude` for each observation so that it could be updated to a more realistic value at a later date.

`HCSS-19293`
: A suggestion to remove the on-board distortion correction by the modified Newton method described in Appendix A.1.

`HCSS-19463`
: We need a way of handling observations (or periods of time) where the overall quality of the attitude determination performed by `calcStrAttitude` is poor. Consistently large values of TASTE are an indication of a problem, even when each individual value is not sufficiently large to reject the fit.

`HCSS-19472`
: The high correlation (between axes) of the measurement errors used by `calcGyroAttitude` means that a generalized least-squares method is required.[46] Although the simplification of decoupling the problem into three weighted least-squares fits probably has negligible effect on the reconstructed attitudes, the consequence is that we have no knowledge of how the errors in the reconstructed attitude are correlated (between axes).

`HCSS-19775`
: Currently, it is still possible that a bad quality attitude measurement may be chosen as the initial reference attitude (see p. 14, footnote 27.)

[46] See p. 19, footnote 35.

A Inversion of distortion correction equations

A.1 Using a modified Newton method

A simple and very effective way of inverting the system of algebraic equations (9), which avoids completely the necessity of having to estimate new sets of coefficients for the distortion correction polynomials, is to set $y_0 = y'$, $z_0 = z'$ and to iterate using:[47]

$$\begin{aligned} y_{i+1} &= y' + y_i - F_1(y_i, z_i; k_0, \ldots, k_7), \\ z_{i+1} &= z' + z_i - F_1(z_i, y_i; h_0, \ldots, h_7). \end{aligned} \tag{45}$$

Writing (9) as

$$\begin{aligned} g_1(y, z) &\equiv F_1(y, z; k_0, \ldots, k_7) - y' = 0, \\ g_2(y, z) &\equiv F_1(z, y; h_0, \ldots, h_7) - z' = 0, \end{aligned} \tag{46}$$

Newton's method gives:

$$\begin{aligned} \begin{pmatrix} y_{i+1} \\ z_{i+1} \end{pmatrix} &= \begin{pmatrix} y_i \\ z_i \end{pmatrix} - \left[\frac{\partial(g_1, g_2)}{\partial(y, z)} \right]^{-1}_{(y_i, z_i)} \begin{pmatrix} g_1(y_i, z_i) \\ g_2(y_i, z_i) \end{pmatrix} \\ &= \begin{pmatrix} y_i \\ z_i \end{pmatrix} - \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1}_{(y_i, z_i)} \begin{pmatrix} F_1(y_i, z_i; k_0, \ldots, k_7) - y' \\ F_1(z_i, y_i; h_0, \ldots, h_7) - z' \end{pmatrix}, \end{aligned} \tag{47}$$

where

$$\begin{aligned} J_{11} &= k_1 - 2\,k_5 y - k_6 z + k_3(3y^2 + z^2) + k_4(5y^4 + 6\,y^2 z^2 + z^4), \\ J_{12} &= k_2 - k_6 y - 2\,k_7 z + 2\,k_3 yz + 4\,k_4 yz(y^2 + z^2), \\ J_{21} &= h_2 - 2\,h_7 y - h_6 z + 2\,h_3 yz + 4\,h_4 yz(y^2 + z^2), \\ J_{22} &= h_1 - h_6 y - 2\,h_5 z + h_3(y^2 + 3\,z^2) + h_4(y^4 + 6y^2 z^2 + 5z^4). \end{aligned} \tag{48}$$

We see therefore that the intuitive scheme (45) corresponds to the approximation $J_{11} = J_{22} = 1$, $J_{12} = J_{21} = 0$. In general, the quadratic convergence exhibited by Newton's method is lost when one approximates the (inverse) Jacobian by a constant matrix and only linear convergence can be expected [10, pp. 109–110]. However, it appears that in the problem under consideration the approximation is sufficiently accurate that little of the quadratic

[47] For the sake of clarity the subscript 'r' on y_r, z_r, y'_r and z'_r has been dropped here.

convergence is lost. That is, it has been found that two iterations of (45) are sufficient to match the accuracy of the current method and that a third iteration reduces the errors to below $0.0001''$.

A.2 Using power series

Berrighi claims that the solution of (1), or equivalently for our purposes (9), will exist in the form of Taylor series and that the "same level of accuracy" can be obtained by truncating these series at degree five [4, p. 84].[48]

In an attempt to justify this, rewrite (9) as:

$$g_i(y, z; y', z') = 0, \quad i = 1, 2.$$

The implicit functions g_1 and g_2 are polynomials and hence analytic in y, z, y' and z'. Equations (48) show the value of the Jacobian determinant at the origin to be

$$\left|\frac{\partial(g_1, g_2)}{\partial(y, z)}\right|_{y=z=0} = |k_1 h_1 - k_2 h_2| \approx |k_1 h_1|.$$

Since this determinant is non-zero, as may be easily verified from the values found for the distortion correction coefficients, and the solution satisfies the conditions

$$g_i(0, 0; k_0, h_0) = 0, \quad i = 1, 2,$$

it follows from the theory of analytic implicit functions [e.g. 13, ch. 6] that y and z may be solved as power series of the form

$$\begin{aligned} y &= \sum_{i,j=0}^{\infty} c_{ij} (y' - k_0)^i (z' - h_0)^j, \\ z &= \sum_{i,j=0}^{\infty} d_{ij} (y' - k_0)^i (z' - h_0)^j, \end{aligned} \tag{49}$$

where the c_{ij} and d_{ij} are real constants and $c_{00} = d_{00} = 0$.

A formal solution may be obtained by the method of undetermined coefficients and Cauchy's method of majorants may then be used to show that

[48] The second statement is understood to mean that the errors incurred in truncating the power series solution at the terms of degree five is negligible compared with the accuracy of the distortion correction model (a system of polynomials of degree five).

these series converge for sufficiently small $y' - k_0$ and $z' - h_0$. However, without further analysis, there is no guarantee that these series converge over the entire range of values corresponding to the star tracker's CCD and Berrighi's claim regarding the level of accuracy that may be achieved by truncating the series at the terms of degree five would seem to be entirely unjustified.

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